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Lec 11:

## Bremsstrahlung:

As we saw before, Conservation of energy and momentum does not allow an isolated charged particle to spontaneously absorb or emit radiation. This conforms with the fact that only an accelerating charged particle can radiate. Emitti[ng] photons from a charged particle is possible when another source of energy and momentum lies nearby (which can lead to acceleration). This could be the Coulomb field of an ion. Under the guise of electron-proton scattering, th[is] is one of the most common manifestations of "Bremsstr[ahlung]" radiation in astronomy.

We have already found an expression for the acceleratio[n] of an electron in the electrostatic field of a high-energ[y] proton or nucleus. Now the roles of the particles are interchanged. Nontheless, the field experienced by the electron in its rest frame is exactly the same as before. We therefore know exactly the acceleration experienced by the electron. We can then use the expression that was derived last time to work out the radiation spectrum of the emitted radiation. Note that the electron motion in its rest frame is non-relativistic (due to short time of interaction). This implies that we can use the relation for angle-integrated distribution found in the dipole approximation. In terms of the Fourier transform of the acceleration, it reads:

$$\frac{dW}{d\omega} = \frac{8\pi e^2}{3c^3} \left| \vec{a}_{(\omega)} \right|^2$$

Now let us find $\vec{a}_{(\omega)}$ in the electron rest frame. We note that $\vec{a}$ is the same in that frame and the lab frame if the electron moves non-relativistically in the lab frame. In the case of relativistic motion, we perform a transformation back to the lab frame to find $\frac{dW}{d\omega}$ there.

$\vec{a}(t)$ has a component parallel to the ion's motion $a_{\parallel}(t)$ and a component that is perpendicular to the direction of motion $a_{\perp}(t)$. These are given by;

Collision parameter

$$a_{\parallel}(t) = \frac{\gamma Z e^2 v t}{m_e [b^2 + (\gamma v t)^2]^{3/2}} , \quad a_{\perp}(t) = \frac{\gamma Z e^2 b}{m_e [b^2 + (\gamma v t)^2]^{3/2}}$$

Here we have used the exact expression for the electric field of a moving ion. We do not need to worry about the magnetic field induced by the ion since electron does not reach relativistic velocities during its interaction, and hence the magnetic force will be negligible.

Going to the frequency domain, we have:

$$a_{\parallel}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\gamma Z e^2 v t}{m_e[b^2 + (\gamma v t)^2]^{3/2}} e^{i\omega t}\, dt$$

$$a_{\perp}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\gamma Z e^2 b}{m_e[b^2 + (\gamma v t)^2]^{3/2}} e^{i\omega t}\, dt$$

These integrals can be written in closed form:

$$a_{\parallel}(\omega) = \frac{1}{2\pi}\frac{Ze^2}{m_e}\frac{1}{\gamma b v} I_1(y) \qquad I_1(y) = 2iy\, K_0(y)$$

$$a_{\perp}(\omega) = \frac{1}{2\pi}\frac{Ze^2}{m_e}\frac{1}{b v} I_2(y) \qquad I_2(y) = 2y\, K_1(y)$$

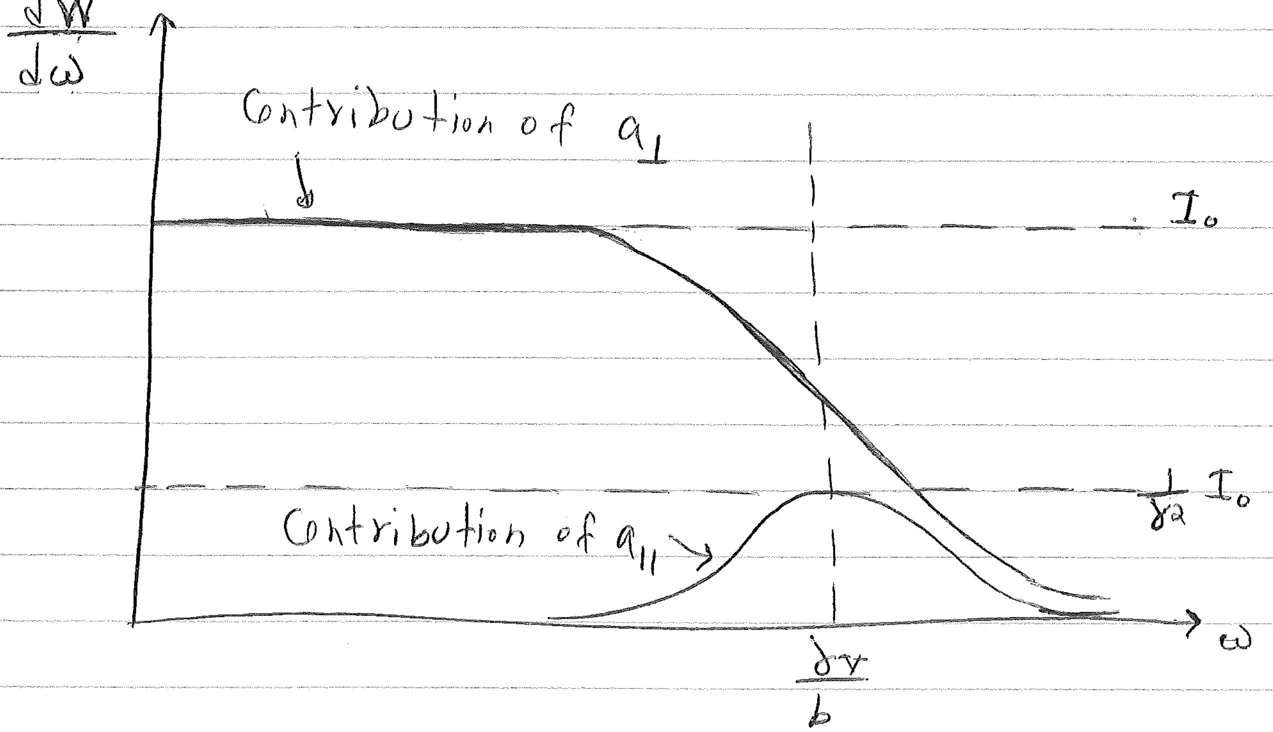
Here $y = \frac{\omega b}{\gamma v}$ and $K_0$, $K_1$ are modified Bessel functions of order zero and one respectively.

Putting things together, we find:

$$\frac{dW}{d\omega} = \frac{8\pi e^2}{3c^3}\left|\vec{a}(\omega)\right|^2 = \frac{8\pi e^2}{3c^3}\left[|a_{\parallel}(\omega)|^2 + |a_{\perp}(\omega)|^2\right] =$$

$$\frac{8 Z^2 e^6}{3\pi c^3 m_e^2}\frac{\omega^2}{\gamma^2 v^2}\left[\frac{1}{\gamma^2}K_0^2\left(\frac{\omega b}{\gamma v}\right) + K_1^2\left(\frac{\omega b}{\gamma v}\right)\right]$$

It is interesting to plot the spectrum, displaying separately the terms that arise from $a_{\parallel}$ and $a_{\perp}$ components:




We note that in the relativistic regime, where $\gamma \gg 1$, the contribution of $a_{\parallel}$ is negligible. Moreover, both term drop exponentially for frequencies larger than $\sim \frac{\gamma v}{b}$. This cut-off can be understood from the fact that the duration of interaction is roughly $\tau = \frac{2b}{\gamma v}$ (in the frame of the electron). Therefore the spread in the frequency domain is expected to be $\Delta\omega \sim \frac{2\pi}{\tau} \sim \frac{\gamma v}{b}$. The exponential cut-off implies that there is little power emitted at frequencies greater than $\omega \approx \frac{\gamma v}{b}$.

It is instructive to study the asymptotic limits of $\frac{dW}{d\omega}$.

At high frequencies, we have:

$$\frac{dW}{d\omega} \approx \frac{4Z^2 e^6}{3c^3 m_e^2} \frac{1}{\gamma v^3} \left[\frac{1}{\gamma^2}+1\right] \exp\left(-\frac{2\omega b}{v\gamma}\right) \qquad \left(\omega \gg \frac{\gamma v}{b}\right)$$

At lowe frequencies, on the other hand, we have:

$$\frac{dW}{d\omega} \approx \frac{8Z^2 e^6}{3\pi c^3 m_e^2} \frac{1}{b^2 v^2} \left[1 - \frac{1}{\gamma^2}\left(\frac{\omega b}{\gamma v}\right)^2 \ln^2\left(\frac{\omega b}{\gamma v}\right)\right] \qquad \left(\omega \ll \frac{\gamma v}{b}\right)$$

In this limit the second term inside the brackets can be ngleeted. The spectrum therefore asymptotes to a constant. This can be understood as follows. As far as low frequencies are concerned, the momentum impulse felt by the electron is a delta-function since the duration of collision is much shorter than the period of these modes. As a result, the spectrum is flat at these frequencies.

Finally, we have to integrate over all relevant collision parameters as we did in the case of ionization. The number density of ions in the electron frame is $\gamma n$, where $n$ is the density in the lab frame. The number of encounters per unit time is $nv$, which results in:

$$\frac{d^2W}{d\omega dt} = \int_{b_{min}}^{b_{max}} 2\pi b \gamma n v \frac{dW}{d\omega}(b)\, db$$

At low frequencies $\left(\omega \lesssim \frac{\gamma v}{b}\right)$, this becomes:

$$\frac{d^2W}{d\omega dt} = \frac{16 Z^2 e^6}{3 c^3 m_e^2} \frac{\gamma n}{v} \ln\left(\frac{b_{max}}{b_{min}}\right)$$

As in the case of ionization, we have to make proper choices of $b_{min}$ and $b_{max}$.

The full answer, which uses full quantum mechanical treatment of the radiation process, is due to Bethe and Heitler.

The key aspect of their result is that the electron cannot give up more than its total kinetic energy in the radiation process. Therefore no photons are radiated with energies greater than $\varepsilon = \hbar\omega = (\gamma - 1) m_e c^2$. The Bethe Heitler formula is:

$$\frac{d^2W}{d\omega dt} = \frac{8}{3} Z^2 \alpha r_e^2 \frac{m_e c^2}{E} \gamma v n \ln\left[\frac{1 + \left(1 - \frac{\varepsilon}{E}\right)^{1/2}}{1 - \left(1 - \frac{\varepsilon}{E}\right)^{1/2}}\right]$$

Here $\alpha \equiv \frac{e^2}{\hbar c} \simeq \frac{1}{137}$ is the fine structure constant and $r_e = \frac{e^2}{m_e c^2}$ is the classical radius of the electron. Also $E = (\gamma - 1) m_e c^2$ is the kinetic energy of the electron and $\varepsilon = \hbar\omega$ is the photon energy.